

BLOWUP FOR THE DAMPED L^2 -CRITICAL NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. We consider the Cauchy problem for the L^2 -critical damped nonlinear Schrödinger equation. We prove existence and stability of finite time blowup dynamics with the log-log blow-up speed for $\|\nabla u(t)\|_{L^2}$.

1. INTRODUCTION

In this paper, we study the blowup of solutions to the Cauchy problem for the L^2 -critical damped nonlinear Schrödinger equations:

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u + iau = 0, (t, x) \in [0, \infty[\times \mathbb{R}^d, d = 1, 2, 3, 4. \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases} \quad (1.1)$$

with initial data $u(0) = u_0 \in H^1(\mathbb{R}^d)$ and where $a > 0$ is the coefficient of friction. Equation (1.1) arises in various areas of nonlinear optics, plasma physics and fluid mechanics. It is known that the Cauchy problem for (1.1) is locally well-posed in $H^1(\mathbb{R}^d)$ (see Kato [6] and also Cazenave [2]): For any $u_0 \in H^1(\mathbb{R}^d)$, there exist $T \in (0, \infty]$ and a unique solution $u(t)$ of (1.1) with $u(0) = u_0$ such that $u \in C([0, T]; H^1(\mathbb{R}^d))$. Moreover, T is the maximal existence time of the solution $u(t)$ in the sense that if $T < \infty$ then $\lim_{t \rightarrow T} \|u(t)\|_{H^1(\mathbb{R}^d)} = \infty$.

Ohta [14] and Tsutsumi [17] studied the supercritical case ($|u|^p u$ with $p > \frac{4}{d}$) and showed that blow-up in finite time can occur, using the virial method. However this method does not seem to apply in the critical case. Therefore, even if numerical simulations suggest the existence of finite time blowup solutions in this case (see Fibich [4]), there does not exist any mathematical proof of blow-up in the critical case. Let us notice that for $a = 0$ (1.1) becomes the L^2 -critical nonlinear Schrödinger equation:

$$\begin{cases} iu_t + \Delta u + |u|^{\frac{4}{d}}u = 0 \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases} \quad (1.2)$$

This equation (1.2) admits a number of symmetries in the energy space H^1 : if $u(t, x)$ is a solution to (1.2) then $\forall \lambda_0 \in \mathbb{R}$, so is $\lambda_0^{\frac{d}{2}} u(\lambda_0 x, \lambda_0^2 t)$.

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Note that the L^2 -norm is left invariant by the scaling symmetry and thus L^2 is the critical space associated with this symmetry.

The evolution of (1.2) admits the following conservation laws in the energy space H^1 :

$$L^2 \text{ norm : } \|u(t, x)\|_{L^2} = \|u(0, x)\|_{L^2} = \|u_0(x)\|_{L^2}.$$

$$\text{Energy : } E(u(t, x)) = \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{d}{4+2d} \|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2} = E(u_0).$$

$$\text{Kinetic momentum : } P(u(t)) = \text{Im} \left(\int \nabla u \bar{u}(t, x) \right) = P(u_0).$$

Special solutions play a fundamental role for the description of the dynamics of (1.2). They are the solitary waves of the form $u(t, x) = \exp(it)Q(x)$, where Q solves:

$$\Delta Q + Q|Q|^{\frac{4}{d}} = Q. \quad (1.3)$$

Equation (1.3) is a standard nonlinear elliptic equation, that possesses a unique positive solution (see [1], [8], [7]).

For $u_0 \in H^1$, a sharp criterion for global existence has been exhibited by Weinstein [18]:

a) For $\|u_0\|_{L^2} < \|Q\|_{L^2}$, the solution of (1.2) is global in H^1 . This follows from the conservation of the energy and the L^2 norm and the sharp Gagliardo-Nirenberg inequality:

$$\forall u \in H^1, E(u) \geq \frac{1}{2} \left(\int |\nabla u|^2 \right) \left(1 - \left(\frac{\int |u|^2}{\int |Q|^2} \right)^{\frac{2}{d}} \right).$$

b) There exists blow-up solutions emanating from initial data $u_0 \in H^1$ with $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$. This follows from the pseudo-conformal symmetry applied to the solitary waves. In the series of papers [9, 16], Merle and Raphael have studied the blowup for the L^2 -critical nonlinear Schrödinger equation (1.2) and have proven the existence of the blowup regime corresponding to the log-log law:

$$\|u(t)\|_{H^1(\mathbb{R}^d)} \sim \left(\frac{\log |\log(T-t)|}{T-t} \right)^{\frac{1}{2}}. \quad (1.4)$$

This regime has the advantage to be stable with respect to H^1 -perturbation and with respect some perturbations of the equation.

Remark 1.1. Based on the works [9, 12] we have the following result: Let u_0 the initial data $\in H^1(\mathbb{R}^d)$ with small super-critical mass:

$$\|Q\|_{L^2} < \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha_0 \quad (1.5)$$

with nonpositive Hamiltonian $E(u_0) < 0$, then the corresponding solution to (1.2) blowup in finite time with the log-log speed.

In the case of (1.1), there does not exist conserved quantities anymore. However, it is easy to prove that if u is a solution of (1.1) then:

$$\|u(t)\|_{L^2} = \exp(-at) \|u_0\|_{L^2}, t \in [0, T), \quad (1.6)$$

$$\frac{d}{dt}E(u(t)) = -a(\|\nabla u\|_{L^2}^2 - \|u\|_{L^{\frac{4}{d}+2}}^{\frac{4}{d}+2}) \quad (1.7)$$

and

$$|P(u(t))| = \exp(-2at)|P(u_0)|, t \in [0, T]. \quad (1.8)$$

In this paper, we will show that:

- (1) if $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$, then the solution of (1.1) is global in H^1 .
- (2) The existence of finite time blowup solutions.

More precisely, we have the following theorem:

Theorem 1.1. *Let u_0 in $H^1(\mathbb{R}^d)$ with $d = 1, 2, 3, 4$:*

- (1) *if $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$ then the solution of (1.1) is global in H^1 .*
- (2) *There exists $\delta_0 > 0$ such that $\forall a > 0$ and $\forall \delta \in]0, \delta_0[$, there exists $u_0 \in H^1$ with $\|u_0\|_{L^2} = \|Q\|_{L^2} + \delta$, such that the solution of (1.1) blows up in finite time in the log-log regime.*

To show the existence of the explosive solutions, we will put us in the log-log regime described by Merle and Raphael.

The global existence will be proved thanks to a L^2 -concentration phenomenon (see Proposition 2.2 in the next section).

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2. L^2 -CONCENTRATION

In this section, we prove assertion (1) of Theorem 1.1 by extending the proof of the L^2 -concentration phenomenon, proved by Ohta and Todorova [14] in the radial case, to the non radial case.

Hmidi and Keraani showed in [5] the L^2 -concentration for the equation (1.2) without the hypothesis of radiality, using the following theorem:

Theorem 2.1. *Let $(v_n)_n$ be a bounded family of $H^1(\mathbb{R}^d)$, such that:*

$$\limsup_{n \rightarrow +\infty} \|\nabla v_n\|_{L^2(\mathbb{R}^d)} \leq M \quad \text{and} \quad \limsup_{n \rightarrow +\infty} \|v_n\|_{L^{\frac{4}{d}+2}} \geq m. \quad (2.1)$$

Then, there exists $(x_n)_n \subset \mathbb{R}^d$ such that:

$$v_n(\cdot + x_n) \rightharpoonup V \quad \text{weakly,}$$

$$\text{with } \|V\|_{L^2(\mathbb{R}^d)} \geq \left(\frac{d}{d+4}\right)^{\frac{d}{4}} \frac{m^{\frac{d}{2}+1}+1}{M^{\frac{d}{2}}} \|Q\|_{L^2(\mathbb{R}^d)}.$$

Now we have the following theorem:

Theorem 2.2. *Assume that $u_0 \in H^1(\mathbb{R}^d)$, and suppose that the solution of (1.1) with $u(0) = u_0$ blows up in finite time $T \in (0, +\infty)$. Then,*

for any function $w(t)$ satisfying $w(t) \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \rightarrow \infty$ as $t \rightarrow T$, there exists $x(t) \in \mathbb{R}^d$ such that, up to a subsequence,

$$\limsup_{t \rightarrow T} \|u(t)\|_{L^2(|x-x(t)| < w(t))} \geq \|Q\|_{L^2(\mathbb{R}^d)}.$$

To show this theorem we shall need the following lemma:

Lemma 2.1. *Let $T \in (0, +\infty)$, and assume that a function $F : [0, T) \rightarrow (0, +\infty)$ is continuous, and $\lim_{t \rightarrow T} F(t) = +\infty$. Then, there exists a sequence $(t_k)_k$ such that $t_k \rightarrow T$ and*

$$\lim_{t_k \rightarrow T} \frac{\int_0^{t_k} F(\tau) d\tau}{F(t_k)} = 0. \quad (2.2)$$

For the proof see [14].

Proof of Theorem 2.2:

By the energy identity (1.7), we have

$$E(u(t)) = E(u_0) - a \int_0^t K(u(\tau)) d\tau, \quad t \in [0, T[. \quad (2.3)$$

Where $K(u(t)) = \|\nabla u\|_{L^2}^2 - \|v\|_{L^{\frac{4}{\frac{4}{d}+2}}}^{\frac{4}{d}+2}$, and by the Gagliardo-Nirenberg inequality and (1.6), we have:

$$\begin{aligned} |K(u(t))| &\leq \|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 + \|u(t)\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}} \\ &\leq \|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 + C \|u(t)\|_{L^2(\mathbb{R}^d)}^{\frac{4}{d}} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 \\ &\leq (1 + C \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4}{d}}) \|\nabla u(t)\|_{L^2(\mathbb{R}^d)}^2 \end{aligned}$$

for all $t \in [0, T[$. Moreover, we have $\lim_{t \rightarrow T} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty$, thus by Lemma 2.1, there exists a sequence $(t_k)_k$ such that $t_k \rightarrow T$ and

$$\lim_{k \rightarrow \infty} \frac{\int_0^{t_k} K(u(\tau)) d\tau}{\|\nabla u(t_k)\|_{L^2(\mathbb{R}^d)}^2} = 0. \quad (2.4)$$

Let

$$\rho(t) = \frac{\|\nabla Q\|_{L^2(\mathbb{R}^d)}}{\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}} \quad \text{and} \quad v(t, x) = \rho^{\frac{d}{2}} u(t, \rho x)$$

and $\rho_k = \rho(t_k)$, $v_k = v(t_k, \cdot)$. The family $(v_k)_k$ satisfies

$$\|v_k\|_{L^2(\mathbb{R}^d)} \leq \|u_0\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|\nabla v_k\|_{L^2(\mathbb{R}^d)} = \|\nabla Q\|_{L^2(\mathbb{R}^d)}.$$

By (2.3) and (2.4), we have

$$E(v_k) = \rho_k^2 E(u_0) - a \rho_k^2 \int_0^{t_k} K(u(\tau)) d\tau \rightarrow 0, \quad (2.5)$$

which yields

$$\|v_k\|_{L^{\frac{d}{d+2}}}^{\frac{d}{d+2}} \rightarrow \frac{d+2}{d} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2. \quad (2.6)$$

The family $(v_k)_k$ satisfies the hypotheses of Theorem 2.1 with

$$m^{\frac{d}{d+2}} = \frac{d+2}{d} \|\nabla Q\|_{L^2(\mathbb{R}^d)}^2 \quad \text{and} \quad M = \|\nabla Q\|_{L^2(\mathbb{R}^d)},$$

thus there exists a family $(x_k)_k \subset \mathbb{R}^d$ and a profile $V \in H^1(\mathbb{R}^d)$ with

$$\|V\|_{L^2(\mathbb{R}^d)} \geq \|Q\|_{L^2(\mathbb{R}^d)}, \text{ such that,}$$

$$\rho_k^{\frac{d}{2}} u(t_k, \rho_k \cdot + x_k) \rightharpoonup V \in H^1 \quad \text{weakly.} \quad (2.7)$$

Using (2.7), $\forall A \geq 0$

$$\liminf_{n \rightarrow +\infty} \int_{B(0,A)} \rho_n^d |u(t_n, \rho_n x + x_n)|^2 dx \geq \int_{B(0,A)} |V|^2 dx,$$

but $\lim_{n \rightarrow +\infty} \frac{w(t_n)}{\rho_n} = +\infty$ thus $\frac{w(t_n)}{\rho_n} > A$, $\rho_n A < w(t_n)$. This gives immediately:

$$\liminf_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq w(t_n)} |u(t_n, x)|^2 dx \geq \int_{|x| \leq A} |V|^2 dx.$$

This it is true for all $A > 0$ thus :

$$\liminf_{t \rightarrow T} \sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq w(t)} |u(t, x)|^2 dx \geq \int Q^2. \quad (2.8)$$

But for every $t \in [0, T[$, $y \mapsto \int_{|x-y| \leq w(t)} |u(t, x)|^2 dx$ is continuous and goes to 0 at infinity, thus the sup is reached in a point $x(t) \in \mathbb{R}^d$, $\sup_{y \in \mathbb{R}^d} \int_{|x-y| \leq w(t)} |u(t, x)|^2 dx = \int_{|x-x(t)| \leq w(t)} |u(t, x)|^2 dx$ and Theorem 2.2 is proved.

Now the part one of Theorem 1.1 is a consequence of Theorem 2.2 and (1.6).

3. STRATEGY OF THE PROOF OF THEOREM 1.1 PART 2.

We look for a solution of (1.1) such that for t close enough to blowup time, we shall have the following decomposition:

$$u(t, x) = \frac{1}{\lambda^{\frac{d}{2}}(t)} (Q_{b(t)} + \epsilon)(t, \frac{x - x(t)}{\lambda(t)}) e^{i\gamma(t)}, \quad (3.1)$$

for some geometrical parameters $(b(t), \lambda(t), x(t), \gamma(t)) \in (0, \infty) \times (0, \infty) \times \mathbb{R}^d \times \mathbb{R}$, here $\lambda(t) \sim \frac{1}{\|\nabla u(t)\|_{L^2}}$, and the profiles Q_b are suitable deformations of Q related to some extra degeneracy of the problem.

Now we take u_0 in H^1 such that u_0 admits the following controls:

(1) Control of the scaling parameter:

$$0 < b(0) \ll 1 \quad \text{and} \quad 0 < \lambda(0) < e^{-e^{\frac{2\pi}{3b(0)}}}. \quad (3.2)$$

(2) L^2 control of the excess of mass:

$$\|\epsilon(0)\|_{L^2} \ll 1. \quad (3.3)$$

(3) H^1 smallness of $\epsilon(0)$:

$$\int |\nabla \epsilon(0)|^2 + \int |\epsilon(0)|^2 e^{-|y|} \leq \Gamma_{b(0)}^{\frac{3}{4}}. \quad (3.4)$$

(4) Control of the energy and momentum:

$$|E(u_0)| \leq \frac{1}{\sqrt{\lambda(0)}} \quad (3.5)$$

$$|P(u_0)| \leq \frac{1}{\sqrt{\lambda(0)}}. \quad (3.6)$$

Remark 3.1. *To prove that there exists u_0 in $H^1(\mathbb{R}^d)$ satisfying (3.2)-(3.6), we take \tilde{u}_0 in $H^1(\mathbb{R}^d)$ an initial data such that the corresponding solution to (1.2) blows up in the log-log regime as described by Merle and Raphael. Then from [13] there exists a time t_0 such that $\tilde{u}(t_0)$ admits a geometrical decomposition:*

$$\tilde{u}(t_0, x) = \frac{1}{\lambda(t_0)} (Q_{b(t_0)} + \epsilon(t_0)) \left(\frac{x - x(t_0)}{\lambda(t_0)} \right) e^{i\gamma(t)}$$

such that (3.2)-(3.4) hold. Moreover by conservation of the Hamiltonian and the Kinetic momentum:

$$|E(\tilde{u}(t_0))| + |P(\tilde{u}(t_0))| = |E(\tilde{u}_0)| + |P(\tilde{u}_0)| \leq \frac{1}{\sqrt{\|\nabla \tilde{u}(t_0)\|_{L^2}}}$$

for t_0 close enough to blowup time, and hence (3.5) and (3.6) hold. We take $u_0 = \tilde{u}(t_0)$.

These conditions will be denoted by *C.I.* Now we have the following theorem:

Theorem 3.1. *Let $u_0 \in H^1$ satisfying C.I, then for $0 < a < a_0$, a_0 small the corresponding solution $u(t)$ of (1.1) blows up in finite time in the log-log regime.*

The set of initial data satisfying C.I is open in H^1 , using the continuity with regard to the initial data and the parameters, we can prove the following corollary (see the proof in section 5):

Corollary 3.1. *Let $u_0 \in H^1$ be an initial data such that the corresponding solution $u(t)$ of (1.2) blows up in the loglog regime. There exist $\beta_0 > 0$ and $a_0 > 0$ such that if $v_0 = u_0 + h_0$, $\|h_0\|_{H^1} \leq \beta_0$ and $a \leq a_0$, the solution $v(t)$ for (1.1) with the initial data v_0 blowup in finite time.*

Remark 3.2. (1) *Combining Theorem 3.1 and Remark 1.1, we deduce that for $u_0 \in H^1(\mathbb{R}^d)$, having negative energy and satisfying (1.5), there exists $a_0 > 0$ such that the corresponding solution of (1.1) blows up in finite time providing $a \leq a_0$.*
 (2) *It is easy to check that if u is a solution of NLS_a (Equation (1.1)), then $\lambda^{\frac{d}{2}}u(\lambda^2 t, \lambda x)$ is a solution of $NLS_{\lambda^2 a}$. Therefore Corollary 3.1 ensures the existence for any $a > 0$ of explosive solutions emanating from an initial data $u_{0,a} \in H^1$, where $u_{0,a}$ satisfies:*

$$\|Q\|_{L^2} < \|u_{0,a}\|_{L^2} = \|u_0\|_{L^2} < \|Q\|_{L^2} + \alpha_0.$$

After the decomposition (3.1) of u , the log-log regime corresponds to the following asymptotic controls

$$b_s \sim C e^{-\frac{\epsilon}{b}}, \quad -\frac{\lambda_s}{\lambda} \sim b \quad (3.7)$$

and

$$\int |\nabla \epsilon|^2 \lesssim e^{-\frac{\epsilon}{b}}, \quad (3.8)$$

where we have introduced the rescaled time $\frac{ds}{dt} = \frac{1}{\lambda^2}$.

In fact, (3.8) is partly a consequence of the preliminary estimate:

$$\int |\nabla \epsilon|^2 \lesssim e^{-\frac{\epsilon}{b}} + \lambda^2 E(t). \quad (3.9)$$

One then observes that in the log-log regime, the integration of the laws (3.7) yields

$$\lambda \sim e^{-e^{\frac{\epsilon}{b}}} \ll e^{-\frac{\epsilon}{b}}, \quad b(t) \rightarrow 0, \quad t \rightarrow T. \quad (3.10)$$

Hence, the term involving the conserved Hamiltonian is asymptotically negligible with respect to the leading order term $e^{-\frac{\epsilon}{b}}$ which drives the decay (3.9) of b . This was a central observation made by Planchon and Raphael in [15]. In fact, any growth of the Hamiltonian algebraically below $\frac{1}{\lambda^2}$ would be enough. In this paper, we will prove that in the log-log regime, the growth of the energy is estimated by:

$$E(u(t)) \lesssim (\log(\lambda(t)))^2. \quad (3.11)$$

We deduce from (3.9) that:

$$\int |\nabla \epsilon|^2 \lesssim e^{-\frac{\epsilon}{b}}. \quad (3.12)$$

An important feature of this estimate of H^1 flavor is that it relies on a flux computation in L^2 . This allows one to recover the asymptotic laws for the geometrical parameters (3.7) and to close the bootstrap estimates of the log-log regime.

This paper is organized as follows. In Sect. 3, 4 and 5, we recall some nonlinear objects involved in the H^1 description of the log-log regime and set up the bootstrap argument, see Proposition 5.2. In Sect. 6, we will control in the bootstrap regime the growth of the energy and momentum, see Lemma 6.1. In Sect. 7 and 8, we close the bootstrap estimates and conclude the proof of Theorem 1.1 (part 2). Finally, the proof of Corollary 3.1 is postponed at the end of Section 8.

4. CHOICE OF THE BLOW UP PROFILE

Let us introduce the rescaled time :

$$s(t) = \int_0^t \frac{d\alpha}{\lambda^2(\alpha)}.$$

It is elementary to check that, whatever the behavior of $u(t)$ is, one always has:

$$s([0, T]) = \mathbb{R}^+.$$

Let us set:

$$v(s, y) = e^{-i\gamma(t)} \lambda^{\frac{d}{2}} u(t, \lambda(t)x + x(t)),$$

where $y = \lambda(t)x + x(t)$, note that:

$$v_s = -i\gamma_s v + \frac{d}{2} \frac{\lambda_s}{\lambda} v + e^{-i\gamma(t)} \lambda^{2+\frac{d}{2}} u_t + \frac{\lambda_s}{\lambda} y \cdot \nabla v + \frac{x_s}{\lambda} \cdot \nabla v.$$

$$\Delta v = e^{-i\gamma(t)} \lambda^{2+\frac{d}{2}} \Delta u(t, \lambda(t)x + x(t)) \text{ and } v|v|^{\frac{4}{d}} = e^{-i\gamma(t)} \lambda^{2+\frac{d}{2}} u|u|^{\frac{4}{d}}.$$

Now $u(t, x)$ solves (1.1) on $[0, T]$ iff $v(s, y)$ solves: $\forall s \geq 0$,

$$iv_s + \Delta v - v + v|v|^{\frac{4}{d}} = i \frac{\lambda_s}{\lambda} \left(\frac{d}{2} v + y \cdot \nabla v \right) + i \frac{x_s}{\lambda} \cdot \nabla v + \tilde{\gamma}_s v, \quad (4.1)$$

where $\tilde{\gamma}_s = -\gamma_s - 1 - ia\lambda^2$, and a is the coefficient of friction. Now $v(s, y) = Q(y) + \epsilon(s, y)$ and we linearize (4.1) close to Q . The obtained system has the form:

$$i\epsilon_s + L\epsilon = i \frac{\lambda_s}{\lambda} \left(\frac{d}{2} Q + x \cdot \nabla Q \right) + \tilde{\gamma}_s Q + i \frac{x_s}{\lambda} \cdot \nabla Q + R(\epsilon), \quad (4.2)$$

where $R(\epsilon)$ is formally quadratic in ϵ , and $L = (L_+, L_-)$ is the matrix linearized operator closed to Q which has components:

$$L_+ = -\Delta + 1 - (1 + \frac{4}{d})Q^{\frac{4}{d}}, \quad L_- = -\Delta + 1 - Q^{\frac{4}{d}}.$$

A standard approach is to think of equation (4.2) in the following way: it is essentially a linear equation forced by terms depending on the law for the geometrical parameters.

Let us observe that the key geometrical parameter is λ which measures the size of the solution. Let us then set

$$b = -\frac{\lambda_s}{\lambda},$$

and study the simpler version of (4.1):

$$iv_s + \Delta v - v + v|v|^{\frac{4}{d}} + ib(\frac{d}{2}v + y \cdot \nabla v) = 0.$$

We look for solutions of the form $v(s, y) = \overline{Q}_{b(s)}(y)$ where the mapping $b \rightarrow Q_b$ and the laws for $b(s)$ are the unknown. We take b uniformly small and $Q_b|_{b=0} = Q$. Now injecting $v(s, y)$ into the equation, we get:

$$i\frac{db}{ds}(\frac{\partial \overline{Q}_b}{\partial b}) + \Delta \overline{Q}_{b(s)} - \overline{Q}_{b(s)} + ib(s)\left(\frac{d}{2}\overline{Q}_{b(s)} + y \cdot \nabla \overline{Q}_{b(s)}\right) + \overline{Q}_{b(s)}|\overline{Q}_{b(s)}|^{\frac{4}{d}} = 0.$$

We set $\overline{P}_{b(s)} = e^{i\frac{b(s)}{4}}|y|^2\overline{Q}_{b(s)}$ and solve:

$$i\frac{db}{ds}(\frac{\partial \overline{P}_b}{\partial b}) + \Delta \overline{P}_{b(s)} - \overline{P}_{b(s)} + \left(\frac{db}{ds} + b^2(s)\right)\frac{|y|^2}{4}\overline{P}_{b(s)} + \overline{P}_{b(s)}|\overline{P}_{b(s)}|^{\frac{4}{d}} = 0. \quad (4.3)$$

Two remarkable solutions to (4.3) can be obtained as follows:

- Take $b(s) = 0$ and $\overline{P}_{b(s)} = Q$, that is the ground state itself.
- Take $b(s) = b$ and $\overline{P}_{b(s)} = \overline{P}$ for some non zero constant b and \overline{P}_b satisfying:

$$\Delta \overline{P}_b - \overline{P}_b + b^2(s)\frac{|y|^2}{4}\overline{P}_b + \overline{P}_b|\overline{P}_b|^{\frac{4}{d}} = 0. \quad (4.4)$$

The solutions to this non linear elliptic equation are those who produce the explicit self similar profiles solutions to this equation:

$$\Delta \overline{Q}_b - \overline{Q}_b + \overline{Q}_b|\overline{Q}_b|^{\frac{4}{d}} + ib(\frac{d}{2}\overline{Q}_b + y \cdot \nabla \overline{Q}_b) = 0. \quad (4.5)$$

A simple way to see this is to recall that we have set $b = -\frac{\lambda_s}{\lambda}$. Hence from $\frac{ds}{dt} = \frac{1}{\lambda^2}$,

$$b = -\frac{\lambda_s}{\lambda} = -\lambda\lambda_t \text{ ie } \lambda(t) = \sqrt{2b(T-t)},$$

which is the scaling law for the blow up speed.

5. SETTING OF THE BOOTSTRAP

In this section, we recall some fundamental nonlinear objects central to the description of the log-log regime. We then set up the bootstrap argument, in the heart of the proof of Theorem 1.2. The conditions C.I will be to initialize the bootstrap.

Based on Propositions 8 and 9 of [11], we claim:

Proposition 5.1. *There exist universal constants $C > 0$, $\eta^* > 0$ such that the following holds true: for all $0 < \eta < \eta^*$, there exist constants $\nu^*(\eta) > 0$, $b^*(\eta) > 0$ going to zero as $\eta \rightarrow 0$ such that for all $|b| < b^*(\eta)$, setting*

$$R_b = \frac{2}{b}\sqrt{1-\eta}, \quad R_b^- = \sqrt{1-\eta}R_b,$$

$B_{R_b} = \{y \in \mathbb{R}^d, |y| \leq R_b\}$, there exists a unique radial solution $\overline{Q}_b \in L^2(B(0, R))$ to

$$\begin{cases} \Delta \overline{Q}_b - \overline{Q}_b + \overline{Q}_b |\overline{Q}_b|^{\frac{4}{d}} + ib(\frac{d}{2}\overline{Q}_b + y \cdot \nabla \overline{Q}_b) = 0, \\ \overline{P}_b = \overline{Q}_b e^{i\frac{b|y|^2}{4}} > 0 \quad \text{in } B_{R_b}, \\ \overline{Q}_b(0) \in (Q(0) - \nu^*(\eta), Q(0) + \nu^*(\eta)), \overline{Q}_b(R_b) = 0. \end{cases} \quad (5.1)$$

Moreover, let ϕ_b be a smooth radially symmetric cut-off function such that $\phi_b(x) = 0$ for $|x| \geq R_b$ and $\phi_b(x) = 1$ for $|x| \leq R_b^-$, $0 \leq \phi_b(x) \leq 1$ and set

$$Q_b(r) = \overline{Q}_b(r)\phi_b(r)$$

then

$$Q_b \rightarrow Q \quad \text{as } b \rightarrow 0$$

in $L^2(\mathbb{R}^d)$, and Q_b satisfies

$$\Delta Q_b - Q_b + Q_b |Q_b|^{\frac{4}{d}} + ib(\frac{d}{2}Q_b + y \cdot \nabla Q_b) = -\Psi_b, \quad (5.2)$$

where $\Psi_b = 2\nabla \phi_b \nabla Q_b + Q_b(\Delta \phi_b) + iQ_b y \cdot \nabla \phi_b + (\phi_b^{1+\frac{4}{d}} - \phi_b)Q_b |Q_b|^{\frac{4}{d}}$, with

$$\text{supp}(\Psi_b) \subset \{R_b^- \leq |y| \leq R_b\} \quad \text{and} \quad |\Psi_b|_{C^1} \leq e^{-\frac{c}{|b|}}.$$

Eventually, Q_b has supercritical mass:

$$\int |Q_b|^2 = \int Q^2 + c_0 b^2 + o(b^2) \quad \text{as } b \rightarrow 0, \quad (5.3)$$

for some universal constant $c_0 > 0$.

The meaning of this proposition is that one can build localized Q_b on the ball B_{R_b} which are a smooth function of b and approximate Q in a very strong way as $b \rightarrow 0$. These profiles satisfy the self similar equation up to an exponentially small term Ψ_b supported around the turning point $\frac{2}{b}$. The proof of this proposition uses standard variational tools in the setting of non linear elliptic problems, and can be found in [11].

Now one can think of making a formal expansion of Q_b in terms of b , and the first term is non zero:

$$\frac{\partial Q_b}{\partial b}|_{b=0} = -\frac{i}{4}|y|^2 Q.$$

However, the energy of Q_b is degenerated in b at all orders:

$$|E(Q_b)| \leq e^{-\frac{C}{|b|}}, \quad (5.4)$$

for some universal constant $C > 0$.

Now given a well-localized function f , we set:

$$f_d = \frac{d}{2}f + y \cdot \nabla f \text{ and } f_{dd} = (f_d)_d.$$

Note that integration by part yields:

$$(f_d, g)_{L^2} = -(g, f_d)_{L^2}.$$

We next introduce the outgoing radiation escaping the soliton core according to the following lemma (see Lemma 15 from [11]):

Lemma 5.1. *(Linear outgoing radiation) There exist universal constants $C > 0$ and $\eta^* > 0$ such that $\forall 0 < \eta < \eta^*$, there exists $b^*(\eta) > 0$ such that $\forall |b| < b^*(\eta)$, the following holds true: there exists a unique radial solution ζ_b to*

$$\begin{cases} \Delta \zeta_b - \zeta_b + ib(\zeta_b)_d = -\Psi_b \\ \int |\nabla \zeta_b|^2 < +\infty. \end{cases} \quad (5.5)$$

Moreover, let

$$\Gamma_b = \lim_{|y| \rightarrow +\infty} |y|^d |\zeta_b(y)|^2, \quad (5.6)$$

then there holds

$$e^{-(1+c\eta)\frac{\pi}{|b|}} \leq \Gamma_b \leq e^{-(1-c\eta)\frac{\pi}{|b|}}. \quad (5.7)$$

We recall that the solution $u(t)$ admits the decomposition :

$$u(t, x) = \frac{1}{\lambda(t)}(Q_{b(t)} + \epsilon)\left(t, \frac{x-x(t)}{\lambda(t)}\right)e^{i\gamma(t)}$$

where the geometrical parameters are uniquely defined through some orthogonality conditions(see later): Let us assume the following uniform controls on $[0, T]$:

- Control of $b(t)$

$$b(t) > 0 \text{ and } b(t) < 10b(0). \quad (5.8)$$

- Control of λ :

$$\lambda(t) \leq e^{-e^{\frac{\pi}{100b(t)}}} \quad (5.9)$$

and the monotonicity of λ :

$$\lambda(t_2) \leq \frac{3}{2}\lambda(t_1), \forall 0 \leq t_1 \leq t_2 \leq T. \quad (5.10)$$

Let $k_0 \leq k_+$ be an integers and $T^+ \in [0, T]$ such that

$$\frac{1}{2^{k_0}} \leq \lambda(0) \leq \frac{1}{2^{k_0-1}}, \frac{1}{2^{k_+}} \leq \lambda(T^+) \leq \frac{1}{2^{k_+-1}} \quad (5.11)$$

and for $k_0 \leq k \leq k_+$, let t_k be a time such that

$$\lambda(t_k) = \frac{1}{2^k}, \quad (5.12)$$

then we assume the control of the doubling time interval:

$$t_{k+1} - t_k \leq k\lambda^2(t_k). \quad (5.13)$$

- control of the excess of mass:

$$\int |\nabla \epsilon(t)|^2 + \int |\epsilon(t)|^2 e^{-|y|} \leq \Gamma_{b(t)}^{\frac{1}{4}}. \quad (5.14)$$

The following proposition ensures that (5.9)-(5.14) determine a trapping region for the flow. We will prove this proposition in **section 7 (Part 7.3)**. \square

Proposition 5.2. *Assuming that (5.8)-(5.14) hold, then the following controls are also true:*

$$b > 0 \text{ and } b(t) < 5b(0). \quad (5.15)$$

$$\lambda(t) \leq e^{-\frac{\pi}{10b(t)}} \quad (5.16)$$

$$\lambda(t_2) \leq \frac{5}{4}\lambda(t_1), \forall 0 \leq t_1 \leq t_2 \leq T \quad (5.17)$$

$$t_{k+1} - t_k \leq \sqrt{k}\lambda^2(t_k) \quad (5.18)$$

$$\int |\nabla \epsilon(t)|^2 + \int |\epsilon(t)|^2 e^{-|y|} \leq \Gamma_{b(t)}^{\frac{2}{3}}. \quad (5.19)$$

6. CONTROL OF THE ENERGY AND THE KINETIC MOMENTUM

We recall the Strichartz estimates. An ordered pair (q, r) is called admissible if $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, $2 < q \leq \infty$. We define the Strichartz norm of functions $u : [0, T] \times \mathbb{R}^d \mapsto \mathbb{C}$ by:

$$\|u\|_{S^0([0, T] \times \mathbb{R}^d)} = \sup_{(q, r) \text{ admissible}} \|u\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)} \quad (6.1)$$

and

$$\|u\|_{S^1([0, T] \times \mathbb{R}^d)} = \sup_{(q, r) \text{ admissible}} \|\nabla u\|_{L_t^q L_x^r([0, T] \times \mathbb{R}^d)} \quad (6.2)$$

We will sometimes abbreviate $S^i([0, T] \times \mathbb{R}^2)$ with S_T^i or $S^i[0, T]$, $i = 1, 2$. Let us denote the Hölder dual exponent of q by q' so that $\frac{1}{q} + \frac{1}{q'} = 1$. The Strichartz estimates may be expressed as:

$$\|u\|_{S_T^0} \lesssim \|u_0\|_{L^2} + \|(i\partial_t + \Delta)u\|_{L_t^{q'} L_x^{r'}} \quad (6.3)$$

where (q, r) is any admissible pair. Now we will derive an estimate on the energy, to check that it remains small with respect to λ^{-2} :

Lemma 6.1. *Assuming that (5.9)-(5.14) hold, then the energy and kinetic momentum are controlled on $[0, T^+]$ by:*

$$|E(u(t))| \lesssim (\log(\lambda(t)))^2, \quad (6.4)$$

$$|P(u(t))| \leq |P(u_0)|. \quad (6.5)$$

To prove this lemma, we shall need the following one:

Lemma 6.2. *Let u be a solution of (1.2) emanating for u_0 in H^1 . Then $u \in C([0, \Delta T], H^1)$ where $\Delta T = \|u_0\|_{L^2}^{\frac{d-4}{d}} \|u_0\|_{H^1}^{-2}$, and we have the following control*

$$\|u\|_{S^0[t, t+\Delta T]} \leq 2 \|u_0\|_{L^2}, \quad \|u\|_{S^1[t, t+\Delta T]} \leq 2 \|u_0\|_{H^1(\mathbb{R}^d)}.$$

proof: For all $v \in S(\mathbb{R}^d)$ we have:

$$\|v\|_{S^0[t, t+\Delta T]} \lesssim \|v(0)\|_{L^2} + \|(i\partial_t + \Delta)v\|_{L_{\Delta T}^{q'} L_x^{r'}},$$

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 < q < \infty, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

In particular,

$$\left\| \int_0^t e^{i(t-s)\Delta} |u|^{\frac{4}{d}} u \right\|_{S_t^1} \lesssim \left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L_t^1 L_x^2}.$$

Using the Hölder inequality we obtain:

$$\left(\int |u|^{\frac{8}{d}} |\nabla u|^2 \right)^{\frac{1}{2}} \leq \left(\int |u|^{\frac{2(4+d)}{d}} \right)^{\frac{2}{4+d}} \left(\int |\nabla u|^{\frac{2(4+d)}{d}} \right)^{\frac{d}{2(4+d)}}.$$

Integrating in time and applying again Hölder inequality we get:

$$\begin{aligned} \left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([t, t+\Delta T])L^2(\mathbb{R}^d)} &\leq \left(\int \left(\int |u|^{\frac{2(4+d)}{d}} dx \right)^{\frac{2}{4+d} \frac{4+d}{4}} dt \right)^{\frac{4}{4+d}} \\ &\quad \times \left(\int \left(\int |\nabla u|^{\frac{2(4+d)}{d}} dx \right)^{\frac{d}{2(4+d)} \frac{4+d}{d}} dt \right)^{\frac{d}{4+d}}. \end{aligned}$$

Thus:

$$\left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([t, t+\Delta T])L^2(\mathbb{R}^d)} \leq \|u\|_{L^{\frac{4+d}{d}}([t, t+\Delta T])L^{\frac{8+2d}{d}}(\mathbb{R}^d)}^{\frac{4}{d}} \|\nabla u\|_{L^{\frac{4+d}{d}}([t, t+\Delta T])L^{\frac{8+2d}{d}}(\mathbb{R}^d)}.$$

But $(\frac{4+d}{d}, \frac{8+2d}{d})$ is admissible, thus we have:

$$\left\| |u|^{\frac{4}{d}} \nabla u \right\|_{L^1([t, t+\Delta T])L^2(\mathbb{R}^d)} \leq \|u\|_{L^{\frac{4+d}{d}}([t, t+\Delta T])L^{\frac{8+2d}{d}}(\mathbb{R}^d)}^{\frac{4}{d}} \|u\|_{S^1[t, t+\Delta T]}.$$

By Sobolev we have:

$$\begin{aligned} \|u\|_{L^{\frac{4+d}{d}}[t, t+\Delta T]L^{\frac{8+2d}{d}}(\mathbb{R}^d)} &\lesssim \|u\|_{L^{\frac{4+d}{d}}([t, t+\Delta T])H^{\frac{2d}{d+4}}(\mathbb{R}^d)} \\ &\leq (\Delta T)^{\frac{d}{d+4}} \|u\|_{L^\infty([t, t+\Delta T])H^{\frac{2d}{d+4}}(\mathbb{R}^d)}. \end{aligned}$$

Now by interpolation we obtain for $d = 1, 2, 3, 4$:

$$\|u\|_{L^{\frac{4+d}{d}}([t, t+\Delta T])L^{\frac{8+2d}{d}}(\mathbb{R}^d)} \leq (\Delta T)^{\frac{d}{d+4}} \|u\|_{L^\infty([t, t+\Delta T])L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{L^\infty([t, t+\Delta T])H^1(\mathbb{R}^d)}^{\frac{2d}{d+4}}.$$

But since according to (1.6), $\|u\|_{L^\infty([t, t+\Delta T])L^2(\mathbb{R}^d)} \leq \|u_0\|_{L^2(\mathbb{R}^d)}$, we finally get:

$$\begin{aligned} \|u\|_{S^1[t, t+\Delta T]} &\leq \|u(t)\|_{H^1(\mathbb{R}^d)} + (\Delta T)^{\frac{d}{d+4}} \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{S^1_{\Delta T}}^{\frac{2d}{d+4}} \|u\|_{S^1[t, t+\Delta T]}, \\ \text{we deduce that for } \Delta T \leq C \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{d-4}{d}} \|u(t)\|_{H^1(\mathbb{R}^d)}^{-2} &: \end{aligned}$$

$$\|u\|_{S^1[t, t+\Delta T]} \leq 2 \|u(t)\|_{H^1(\mathbb{R}^d)}.$$

In the same way, $\|u\|_{S^0[t, t+\Delta T]} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)} + (\Delta T)^{\frac{d}{d+4}} \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{S^1[t, t+\Delta T]}^{2\frac{d}{d+4}} \|u\|_{S^0[t, t+\Delta T]}$, but $(\Delta T)^{\frac{d}{d+4}} \|u_0\|_{L^2(\mathbb{R}^d)}^{\frac{4-d}{d+4}} \|u\|_{S^1[t, t+\Delta T]}^{2\frac{d}{d+4}} \leq \frac{1}{2}$ we obtain that:

$$\|u\|_{S^0[t, t+\Delta T]} \leq 2 \|u_0\|_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|u\|_{S^1[t, t+\Delta T]} \leq 2 \|u_0\|_{H^1(\mathbb{R}^d)}.$$

Let us return to the proof of the lemma 6.1:

According to (5.18) each interval $[t_k, t_{k+1}]$, can be divided into k intervals, $[\tau_k^j, \tau_k^{j+1}]$ such that the estimates of the previous lemma are true.

From (1.7), we thus deduce that:

$$\begin{aligned} \int_{t_k}^{t_{k+1}} \frac{d}{dt} E(u(t)) &\leq \int_{t_k}^{t_{k+1}} \int |u|^{\frac{4}{d}+2} \\ &\leq \sum_{j=1}^k \int_{\tau_k^j}^{\tau_k^{j+1}} \int |u|^{\frac{4}{d}+2} \\ &\leq \sum_{j=1}^k \|u\|_{S^0_{[\tau_k^j, \tau_k^{j+1}]}}^{\frac{4}{d}+2}. \end{aligned}$$

Since $(\frac{4}{d}+2, \frac{4}{d}+2)$ is admissible. Using that $\|u\|_{S^0[t, t+\Delta T]} \leq 2 \|u_0\|_{L^2(\mathbb{R}^d)}$ independantly of t , we obtain finally :

$$\int_{t_k}^{t_{k+1}} \frac{d}{dt} E(u(t)) \lesssim k.$$

Summing from k_0 to k_+ , we obtain

$$\int_0^{T^+} \frac{d}{dt} E(u(t)) \lesssim |\log(\lambda(T^+))|^2.$$

But $E(u(T^+)) = E(u(0)) + \int_0^T \frac{d}{ds} E(u(s)) ds$, since $|E(u(0))| \leq \frac{1}{\sqrt{\lambda(0)}}$ then we obtain $|E(u(T^+))| \lesssim |\log(\lambda(T^+))|^2$. This completes the proof of (6.4).

Now (6.5) follow directly from (1.8). For sake of completness, let us prove (1.8):

Suppose first that $u(t)$ is very regular (for example $D(\mathbb{R}^d)$)

$$\begin{aligned} \frac{d}{dt} P(u(t)) &= \text{Im} \left(\int \bar{u}_t \nabla u dx + \int \bar{u} \nabla u_t \right) \\ &= \text{Im} \left(\int \bar{u}_t \nabla u dx - \int u_t \nabla \bar{u} \right) \\ &= \text{Im}(2i \text{Im} \int \bar{u}_t \nabla u) \\ &= 2 \text{Im} \int \bar{u}_t \nabla u dx = 2 \text{Im}(i \int i \bar{u}_t \nabla u) = 2 \text{Re} \int i \bar{u}_t \nabla u \\ &= 2 \text{Re} \left(\int (-\Delta \bar{u} - |u|^{\frac{4}{d}} \bar{u} + i a \bar{u}) \nabla u dx \right) \\ &= -2 \text{Re} \left(\int \Delta \bar{u} \nabla u \right) - 2 \text{Re} \int |u|^{\frac{4}{d}} \bar{u} \nabla u + 2a \text{Re} \int i \bar{u} \nabla u. \end{aligned}$$

It is easy to prove that: $-2 \text{Re} \left(\int \Delta \bar{u} \nabla u \right) = \int \nabla |\nabla u|^2$, and $-2 \text{Re} \int |u|^{\frac{4}{d}} \bar{u} \nabla u = -\frac{d}{d+2} \int \nabla(|u|^{\frac{4}{d}+2})$.

But $2aRe \int i\bar{u}\nabla u = -2aIm \int \bar{u}\nabla u = -2aP(u(t))$ we obtain:

$$\frac{d}{dt}P(u(t)) = \int \nabla(|\nabla u|^2 - \frac{d}{d+2}|u|^{\frac{4}{d}+2}) - 2aP(u(t)), \text{ But } \int \left(\nabla(|\nabla u|^2 - \frac{d}{d+2}|u|^{\frac{4}{d}+2}) \right) = 0, \text{ we obtain finally:}$$

$$\frac{d}{dt}P(u(t)) = -2ap(u(t)) \text{ and } P(u(t)) = e^{-2at}P(u_0).$$

Now we take u_0 in $H^1(\mathbb{R}^d)$, u_0 is the limit of a sequence (u_{0n}) in $D(\mathbb{R}^d)$, for each u_{0n} we denote by u_n the solution of (1.1) such that $u_n(0) = u_{0n}$, we have $P(u_n(t)) = e^{-2at}P(u_{0n})$, but $u_0 \mapsto u$ is continuous from $H^1(\mathbb{R}^d)$ to $C([0, T], H^1(\mathbb{R}^d))$ by passing to limit we obtain $P(u(t)) = e^{-2at}P(u(0))$. \square

7. BOOTING THE LOG-LOG REGIME

Now we are going to prove the Lemma 5.2:

First of all we are going to prove the smallness of the L^2 norm of $\epsilon(t)$:

Lemma 7.1. *There exist $\alpha_0 \ll 1$, such that $\forall t \in [0, T]$, $\|\epsilon(t)\|_{L^2(\mathbb{R}^d)} < \alpha_0$.*

Proof: From (3.3) we have $\|u_0\|_{L^2} < \|Q\|_{L^2} + \gamma_0$, with γ_0 very small. By (1.6)

$$\begin{aligned} \|u_0\|_{L^2(\mathbb{R}^d)}^2 &\geq \|u(t)\|_{L^2(\mathbb{R}^d)}^2 \\ &= \|Q_b + \epsilon\|_{L^2(\mathbb{R}^d)}^2 \\ &= \|Q_b + \epsilon\|_{L^2(B(0,R))}^2 + \|Q_b + \epsilon\|_{L^2(\mathbb{R}^d \setminus B(0,R))}^2 \\ \|Q_b + \epsilon\|_{L^2(B(0,R))} &\geq \|Q_b\|_{L^2(B(0,R))} - \|\epsilon\|_{L^2(B(0,R))} \\ &\geq \|Q\|_{L^2(\mathbb{R}^d)} - \|Q_b - Q\|_{L^2(\mathbb{R}^d)} \\ &\quad - \|Q_b\|_{L^2(\mathbb{R}^d \setminus B(0,R))} - \|\epsilon\|_{L^2(B(0,R))} \end{aligned}$$

From (5.14), we have $\|\epsilon\|_{L^2(B(0,R))} < \beta$, where β is very small. Moreover $Q_b \rightarrow Q$ in $L^2(\mathbb{R}^d)$, and $\|Q_b\|_{L^2(\mathbb{R}^d \setminus B(0,R))} \leq \|Q_b - Q\|_{L^2(\mathbb{R}^d)} + \|Q\|_{L^2(\mathbb{R}^d \setminus B(0,R))}$, where $\|Q\|_{L^2(\mathbb{R}^d \setminus B(0,R))} \rightarrow 0$ as $R \rightarrow \infty$.

Therefore:

$$\|Q_b + \epsilon\|_{L^2(B(0,R))} \geq \|u_0\|_{L^2(\mathbb{R}^d)} - \gamma_0 - \beta - \delta,$$

and

$$\begin{aligned} \|Q_b + \epsilon\|_{L^2(\mathbb{R}^d \setminus B(0,R))} &\geq \|\epsilon\|_{L^2(\mathbb{R}^d \setminus B(0,R))} - \|Q_b\|_{L^2(\mathbb{R}^d \setminus B(0,R))} \\ &\geq \|\epsilon\|_{L^2(\mathbb{R}^d \setminus B(0,R))} - \delta, \end{aligned}$$

where $\delta \rightarrow 0$ as $b \rightarrow 0$ and $R \rightarrow \infty$. We obtain finally:

$$\|u_0\|_{L^2(\mathbb{R}^d)}^2 \geq \|\epsilon\|_{L^2(\mathbb{R}^d \setminus B(0,R))}^2 + \|u_0\|_{L^2(\mathbb{R}^d)}^2 - \alpha_0^2,$$

where $\alpha_0 \rightarrow 0$ as $\gamma_0 \rightarrow 0$.

This completes the proof. \square

7.1. Control of the geometrical parameters. Let us now write down the equation satisfied by ϵ in rescaled variables. To simplify the notations, we note

$$Q_b = \Sigma + i\Theta, \quad \epsilon = \epsilon_1 + i\epsilon_2 \text{ and } \Psi_b = Re(\Psi) + iIm(\Psi),$$

in terms of real and imaginary parts.

We have: $\forall s \in \mathbb{R}^+, \forall y \in \mathbb{R}^d$,

$$\begin{aligned} b_s \frac{\partial \Sigma}{\partial b} + \partial_s \epsilon_1 - M_-(\epsilon) + b(\epsilon_1)_d &= \left(\frac{\lambda_s}{\lambda} + b \right) \Sigma_d + \tilde{\gamma}_s \Theta + \frac{x_s}{\lambda} \cdot \nabla \Sigma \\ &+ \left(\frac{\lambda_s}{\lambda} + b \right) (\epsilon_1)_d + \tilde{\gamma}_s \epsilon_2 + \frac{x_s}{\lambda} \cdot \nabla \epsilon_1 \\ &+ Im(\Psi) - R_2(\epsilon). \end{aligned} \quad (7.1)$$

$$\begin{aligned} b_s \frac{\partial \Theta}{\partial b} + \partial_s \epsilon_2 + M_+(\epsilon) + b(\epsilon_2)_d &= \left(\frac{\lambda_s}{\lambda} + b \right) \Theta_d - \tilde{\gamma}_s \Sigma + \frac{x_s}{\lambda} \cdot \nabla \Theta \\ &+ \left(\frac{\lambda_s}{\lambda} + b \right) (\epsilon_2)_d - \tilde{\gamma}_s \epsilon_1 + \frac{x_s}{\lambda} \cdot \nabla \epsilon_2 \\ &- Re(\Psi) + R_1(\epsilon). \end{aligned} \quad (7.2)$$

With $\tilde{\gamma}_s(s) = -1 - \gamma_s(s) - ia\lambda^2$. The linear operator close to Q_b is now

a deformation of the linear operator L close to Q and is $M = (M_+, M_-)$ with

$$\begin{aligned} M_+(\epsilon) &= -\Delta \epsilon_1 + \epsilon_1 - \left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_1 - \left(\frac{4\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \right) \epsilon_2, \\ M_-(\epsilon) &= -\Delta \epsilon_2 + \epsilon_2 - \left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_2 - \left(\frac{4\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \right) \epsilon_1. \end{aligned}$$

The formally quadratic in ϵ interaction terms are:

$$\begin{aligned} R_1(\epsilon) &= (\epsilon_1 + \Sigma) |\epsilon + Q_b|^{\frac{4}{d}} - \Sigma |Q_b|^{\frac{4}{d}} - \left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_1 - \left(\frac{4\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \right) \epsilon_2, \\ R_2(\epsilon) &= (\epsilon_2 + \Sigma) |\epsilon + Q_b|^{\frac{4}{d}} - \Sigma |Q_b|^{\frac{4}{d}} - \left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_2 - \left(\frac{4\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \right) \epsilon_1. \end{aligned}$$

We note $s(0)$ by s_0 and $s(T^+)$ by s^+ , now we have the following lemma:

Lemma 7.2. *(Control of the geometrical parameters) For all $s \in [s_0, s^+]$, there holds:*

- *Estimates induced by the control of energy and momentum:*

$$\begin{aligned} &|2(\epsilon_1, \Sigma + b\Theta_d - Re(\Psi_b)) + 2(\epsilon_2, \Theta - b\Sigma_d - Im(\Psi_b))| \\ &\leq \delta_0 \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right)^{\frac{1}{2}} + \Gamma_{b(s)}^{1-cn} + \lambda^2 |E(u(t))|. \end{aligned} \quad (7.3)$$

$$|(\epsilon_2, \nabla Q)| \leq \delta_0 \|\nabla \epsilon(s)\|_{L^2(\mathbb{R}^d)} + \lambda |P(u)|. \quad (7.4)$$

• *Estimates on the modulation parameters:*

$$\left| \frac{\lambda_s}{\lambda} + b \right| + |b_s| \leq C \left(\int |\nabla \epsilon(s)|^2 + \int |\epsilon(s)|^2 e^{-|y|} \right)^{\frac{1}{2}} + \Gamma_{b(s)}^{1-c\eta}. \quad (7.5)$$

$$\left| \tilde{\gamma}_s - \frac{(\epsilon_1, L_+ Q_{dd})}{\|Q_d\|_{L^2}^2} \right| + \left| \frac{x_s}{\lambda} \right| \leq \delta_0 \left(\int |\epsilon(s)|^2 e^{-|y|} \right)^{\frac{1}{2}} + \Gamma_{b(s)}^{1-c\eta}. \quad (7.6)$$

Here δ_0 is a small constant $\delta_0 \ll 1$.

We will need the following lemma (for the proof see [10]).

Lemma 7.3. *(Control of nonlinear interactions). Let $P(y)$ a polynomial and integers $0 \leq k \leq 3$, $0 \leq l \leq 1$, $0 \leq m \leq 2$, then for some function $\delta(\alpha_0) \rightarrow 0$ as $\alpha_0 \rightarrow 0$,*

- $\left| (\epsilon, P(y) \frac{d^k}{dy^k} Q_b(y)) \right| \leq C_{P,k} \left(\int |\epsilon(s)|^2 e^{-|y|} \right)^{\frac{1}{2}},$
- $\left| (\epsilon, P(y) \frac{d^k}{dy^k} (Q_b(y) - Q(y))) \right| \leq \delta(\alpha_0) \left(\int |\epsilon(s)|^2 e^{-|y|} \right)^{\frac{1}{2}},$
- $\int |\epsilon| \left| P(y) \frac{d^m}{dy^m} \frac{\partial Q_b}{\partial b} \right| \leq \left(\int |\epsilon(s)|^2 e^{-|y|} + \int |\nabla \epsilon|^2 dy \right)^{\frac{1}{2}},$
- $\left| (R(\epsilon), P(y) \frac{d^k}{dy^k} Q_b(y)) \right| \leq C \left(\int |\epsilon(s)|^2 e^{-|y|} dy + \int |\nabla \epsilon|^2 dy \right)^{\frac{1}{2}},$
- $\int |F(\epsilon)| + \left| (\tilde{R}_1(\epsilon), \Sigma_d) \right| + \left| (\tilde{R}_2(\epsilon), \Theta_d) \right| \leq \delta(\alpha_0) \left(\int |\epsilon(s)|^2 e^{-|y|} dy + \int |\nabla \epsilon|^2 dy \right)^{\frac{1}{2}},$
- $\left| \left(P(y) \frac{d^l}{dy^l} \Psi, \frac{d^k}{dy^k} Q_b(y) \right) \right| + \left| (\epsilon, P(y) \frac{d^l}{dy^l} \Psi) \right| \leq e^{-\frac{C}{|b|}},$
- $\left| \left(\frac{\partial Q_b}{\partial b}, P(y) \frac{d^k}{dy^k} Q_b(y) \right) + \left(i \frac{|y|^2}{4} Q, P(y) \frac{d^k}{dy^k} Q(y) \right) \right| \leq \delta(\alpha_0).$

Proof of Lemma 7.2: To prove (7.3), we rewrite the expression of energy in the ϵ variable ($\epsilon = e^{i\gamma(t)} \lambda^{\frac{d}{2}}(t) u(t, \lambda(t)x + x(t)) - Q_b$):

$$\begin{aligned} & 2(\epsilon_1, \Sigma + b\Theta_d - Re(\Psi)) + 2(\epsilon_2, \Theta - b\Sigma_d - Im(\Psi)) \\ &= 2E(Q_b) - 2\lambda^2 E(u(t)) \\ &+ \int |\nabla \epsilon|^2 - \int \left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_1^2 \\ &- \int \left(\frac{4\Theta^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_2^2 \\ &- 8 \int \frac{\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \epsilon_1 \epsilon_2 - \frac{2}{2 + \frac{4}{d}} \int F(\epsilon). \end{aligned} \quad (7.7)$$

With

$$\begin{aligned}
F(\epsilon) &= |\epsilon + Q_b|^{\frac{4}{d}+2} - |Q_b|^{\frac{4}{d}+2} - \left(\frac{4}{d} + 2\right) \frac{|Q_b|^{\frac{4}{d}+2}}{|Q_b|^2} (\Sigma\epsilon_1 + \Theta\epsilon_2) \\
&\quad - \epsilon_1^2 \frac{|Q_b|^{\frac{4}{d}+2}}{|Q_b|^4} \left(\left(\frac{2}{d} + 1\right) \left(\frac{4}{d} + 1\right) \Sigma^2 + \left(\frac{2}{d} + 1\right) \Theta^2 \right) \\
&\quad - \epsilon_2^2 \frac{|Q_b|^{\frac{4}{d}+2}}{|Q_b|^4} \left(\left(\frac{2}{d} + 1\right) \left(\frac{4}{d} + 1\right) \Theta^2 + \left(\frac{2}{d} + 1\right) \Sigma^2 \right) \\
&\quad - \epsilon_1 \epsilon_2 \frac{|Q_b|^{\frac{4}{d}+2}}{|Q_b|^4} \frac{8}{d} \left(\frac{2}{d} + 1\right) \Sigma \Theta.
\end{aligned}$$

(7.3) then follows from Lemma 7.3 (we estimate the terms in (7.7) using Lemma 7.3, and we obtain (7.3)).

Now to prove (7.4), we rewrite the expression of the moment in the ϵ variable:

$$\begin{aligned}
P(u(t)) &= \text{Im} \int (\nabla u \bar{u}) = \frac{1}{\lambda} \text{Im} \left(\int (\nabla \epsilon + \nabla Q_b) \overline{(\epsilon + Q_b)} \right) \\
&= \frac{1}{\lambda} \left(\text{Im} \left(\int \nabla \epsilon \bar{\epsilon} \right) - 2(\epsilon_2, \nabla \Sigma) + 2(\epsilon_1, \nabla \Theta) \right),
\end{aligned}$$

so that

$$2(\epsilon_2, \nabla \Sigma) = 2(\epsilon_1, \nabla \Theta) + \text{Im} \left(\int \nabla \epsilon \bar{\epsilon} \right) - \lambda P(u(t)).$$

From $\Theta_{b=0} = 0$ and the smallness of the L^2 norm of $\epsilon(t)$ and the control(1.8) of the momentum we obtain (7.4).

The prove (7.5)-(7.6), it suffices to follow the proof of Lemma 3 in [10]. \square

Now let

$$\begin{aligned}
\tilde{R}_1(\epsilon) &= R_1(\epsilon) - \epsilon_1^2 \frac{|Q_b|^{\frac{4}{d}}}{|Q_b|^4} \left(\frac{2}{d} \left(\frac{4}{d} + 1\right) \Sigma^3 + \frac{6}{d} \Sigma \Theta^2 \right) \\
&\quad - \epsilon_2^2 \frac{|Q_b|^{\frac{4}{d}}}{|Q_b|^4} \left(\frac{2}{d} \Sigma^3 + \frac{2}{d} \left(\frac{4}{d} - 1\right) \Sigma \Theta^2 \right) \\
&\quad - \frac{4}{d} \frac{|Q_b|^{\frac{4}{d}}}{|Q_b|^4} \epsilon_1 \epsilon_2 \left(\left(\frac{4}{d} - 1\right) \Sigma^2 \Theta + \Theta^3 \right), \tag{7.8}
\end{aligned}$$

$$\begin{aligned}
\tilde{R}_2(\epsilon) = R_2(\epsilon) - \epsilon_2^2 \frac{|Q_b|^{\frac{4}{d}}}{|Q_b|^4} & \left(\frac{2}{d} \left(\frac{4}{d} + 1 \right) \Theta^3 + \frac{6}{d} \Sigma^2 \Theta \right) \\
& - \epsilon_1^2 \frac{|Q_b|^{\frac{4}{d}}}{|Q_b|^4} \left(\frac{2}{d} \Theta^3 + \frac{2}{d} \left(\frac{4}{d} - 1 \right) \Sigma^2 \Theta \right) \\
& - \frac{4}{d} \frac{|Q_b|^{\frac{4}{d}}}{|Q_b|^4} \epsilon_1 \epsilon_2 \left(\left(\frac{4}{d} - 1 \right) \Sigma \Theta^2 + \Sigma^3 \right). \tag{7.9}
\end{aligned}$$

We define the two real Shrödinger operators:

$$L_1 = -\Delta + \frac{2}{d} \left(\frac{4}{d} + 1 \right) Q^{\frac{4}{d}-1} y \cdot \nabla Q, \quad L_2 = -\Delta + \frac{2}{d} Q^{\frac{4}{d}-1} y \cdot \nabla Q.$$

To show the explosion, we will need to control b_s . Note that our continuous functions $(\lambda, \gamma, x(t), b)$ such that:

$$\epsilon = e^{i\gamma(t)} \lambda^{\frac{d}{2}}(t) u(t, \lambda(t)x + x(t)) - Q_b$$

satisfy the following conditions of orthogonality:

$$(\epsilon_1(t), \Sigma_d) + (\epsilon_2(t), \Theta_d) = 0, \tag{7.10}$$

$$(\epsilon_1(t), y\Sigma) + (\epsilon_2(t), y\Theta) = 0, \tag{7.11}$$

$$-(\epsilon_1(t), \Theta_{dd}) + (\epsilon_2(t), \Sigma_{dd}) = 0, \tag{7.12}$$

$$-(\epsilon_1(t), \Theta_d) + (\epsilon_2(t), \Sigma_d) = 0. \tag{7.13}$$

For the proof of these conditions see Lemma 2 in [10], the proof is based on the implicit function theorem using that $(Q_b)_{b=0} = Q$ and $(\frac{\partial Q_b}{\partial b})_{b=0} = -i\frac{|y|^2}{4}Q$.

Now we have the following one:

Proposition 7.1. *There exist $\delta_0 > 0$, $C > 0$ and $0 < \beta < 2$ such that:*

$$b_s \geq \delta_0 \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right) - e^{-\frac{C}{b}} - \lambda^\beta(s). \tag{7.14}$$

Proof: To prove this proposition, it suffices to follow the proof of Proposition 3 in [10] and used the control of the energy. \square

We will need to refine Q_b because Q_b is not an exact sel-similar solution. The basic idea is that the profile $Q_b + \zeta_b$ should be a better approximation of the solution. The problem is now that ζ_b is indeed in \dot{H}^1 , but not in L^2 , and we then are not able to estimate the main interaction terms. We therefore introduce a cut version of the radiation: leave a radial cutoff function: $\chi_A(r) = \chi(\frac{r}{A})$ with $\chi(r) = 1$ for $0 \leq r \leq 1$ and $\chi(r) = 0$ for $r \geq 2$.

The choice of the parameter $A(t)$ is a crucial issue in our analysis, and is roughly based on two constraints: we want A to be large in order first to enter the radiative zone, i.e., $\frac{2}{b} \ll A$, and to ensure the slowest possible variations of the L^2 -norm in the zone $|y| \geq A$. But we also

want A not too large, in particular to keep a good control over local L^2 -terms of the form $\int_{|y| \leq A} |\epsilon|^2$.

A choice which balances these two constraints is:

$$A = A(t) = e^{2l \frac{1}{b(t)}} \text{ so that } \Gamma_b^{-\frac{l}{2}} \leq A \leq \Gamma_b^{-\frac{3l}{2}},$$

for some parameter $l > 0$ small enough to be chosen later and which depends on η . Now let

$$\tilde{\zeta} = \chi\left(\frac{r}{A}\right)\zeta_b,$$

Observe that $\tilde{\zeta}$ is now a small Schwartz function thanks to the A localization. we next consider the new variable

$$\tilde{\epsilon} = \epsilon - \tilde{\zeta}, \quad (7.15)$$

$\tilde{\zeta}_b$ still satisfies the size estimates of Lemma 2.1 and is moreover in L^2 with an estimate

$$\int |\tilde{\zeta}|^2 \leq \Gamma_b^{1-C\eta}. \quad (7.16)$$

The equation satisfied by $\tilde{\zeta}$ is now

$$\Delta \tilde{\zeta} - \tilde{\zeta} + ib(\tilde{\zeta})_d = \Psi_b + F$$

with

$$F = (\Delta \chi_A)\zeta_b + 2\nabla \chi_A \cdot \nabla \zeta_b + iby \cdot \nabla \chi_A \zeta_b. \quad (7.17)$$

Now we have the following lemma (see Lemma 4.4 in [3] for further details):

Lemma 7.4. (*Virial dispersion in the radiative regime*) *There exist a constants $\delta_1 > 0$, $C > 0$ and $\alpha > 0$ such that:*

$$\begin{aligned} (f_1(s))_s &\geq \delta_1 \left(\int |\nabla \tilde{\epsilon}|^2 + \int |\epsilon(s)|^2 e^{-|y|} + \Gamma_b \right) \\ &\quad - \frac{1}{\delta_1} \int_A^{2A} |\epsilon|^2 - C\lambda^2 E(u(t)). \end{aligned} \quad (7.18)$$

with

$$f_1(s) = \frac{b}{4} \|yQ_b\|_{L^2}^2 + \frac{1}{2} \text{Im} \left(\int (y \cdot \nabla \tilde{\epsilon}) \bar{\tilde{\zeta}} \right) + (\epsilon_2, \Lambda \tilde{\zeta}_{re}) - (\epsilon_1, \Lambda \tilde{\zeta}_{im}). \quad (7.19)$$

We now need to control the term $\int_A^{2A} |\epsilon|^2$ in (7.18). This is achieved by computing the flux of L^2 -norm escaping the radiative zone. We introduce a radial nonnegative cut off function $\phi(r)$ such that $\phi(r) = 0$ for $r \leq \frac{1}{2}$, $\phi(r) = 1$ for $r \geq 3$, $\frac{1}{4} \leq \phi'(r) \leq \frac{1}{2}$ for $1 \leq r \leq 2$, $\phi'(r) \geq 0$. We then set

$$\phi_A(s, r) = \phi\left(\frac{r}{A(s)}\right).$$

Moreover, we restrict the freedom on the choice of the parameters (η, l) by assuming $l > C\eta$. We have:

$$\begin{cases} \phi_A(r) = 0 & \text{for } 0 \leq r \leq \frac{A}{2}, \\ \frac{1}{4A} \leq \phi'_A(r) \leq \frac{1}{2A} & \text{for } A \leq r \leq 2A, \\ \phi_A(r) = 1 & \text{for } r \geq 3A \\ \phi'_A(r) \geq 0, 0 \leq \phi_A(r) \leq 1. \end{cases}$$

We now claim the following dispersive control at infinity in space (see Lemma 7 in [13] for the proof):

Lemma 7.5. (*L^2 dispersion at infinity in space*). *For some universal constant $C > 0$, if s large enough:*

$$\left(\int \phi_A |\epsilon|^2 \right)_s \geq \frac{b}{400} \int_A^{2A} |\epsilon|^2 - \Gamma_b^{1+Cl} - \Gamma_b^{\frac{1}{2}} \int |\nabla \epsilon|^2 - \frac{\lambda^2}{b^2} E(u(t)). \quad (7.20)$$

Note that $\lambda \leq e^{-e^{\frac{C}{b}}}$ thus $\frac{\lambda^2}{b^2} E(u(t)) \leq \lambda^\beta$ with $0 < \beta < 2$ close to 2, thus the last term in this estimation is small with respect to λ .

7.2. L^2 -dispersive constraint on the solution. In this subsection, we derive the dispersive estimate needed for the proof of the blowup. The virial estimate (7.18) corresponds to nonlinear interactions on compact sets. The L^2 linear estimate (7.20) measures the interactions with the linear dynamic at infinity. We now couple these two facts through the smallness of the L^2 -norm, which is a global information in space.

Proposition 7.2. *For some universal constant $C > 0$ and for $s \geq 0$, the following holds:*

$$(\mathfrak{S})_s \leq -Cb \left(\Gamma_b + \int |\nabla \tilde{\epsilon}|^2 + \int |\tilde{\epsilon}|^2 e^{-|y|} + \int_A^{2A} |\epsilon|^2 \right) + C \frac{\lambda^2}{b^2} E(u(t)). \quad (7.21)$$

with:

$$\begin{aligned} \mathfrak{S}(s) = & \left(\int |Q_b|^2 - \int |Q|^2 \right) + 2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) + \int (1 - \phi_A) |\epsilon|^2 \\ & - \frac{\delta_1}{800} \left(b \tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) dv + b((\epsilon_2, \Lambda \tilde{\zeta}_{re}) - (\epsilon_1, \Lambda \tilde{\zeta}_{im})) \right). \end{aligned} \quad (7.22)$$

Here $c > 0$ denotes some small enough universal constant and:

$$\tilde{f}_1(b) = \frac{b}{4} |y Q_b|_2^2 + \frac{1}{2} \text{Im} \left(\int (y \cdot \nabla \tilde{\zeta}) \bar{\tilde{\zeta}} \right). \quad (7.23)$$

Remark 7.1. *Here the range of parameters is more restricted and yields: there exist $\eta^*, l^*, C_0 > 0$ such that $\forall 0 < \eta < \eta^*, \forall 0 < l < l^*$ such that $l > C_0 \eta$, there exists $b^*(\eta, l)$ such that $\forall |b| \leq b^*(\eta, l)$, the estimates of this proposition hold with universal constants.*

Remark 7.2. *The gain is that we now have a Lyapunov function \mathfrak{S} in H^1 . Remark that in a regime when ϵ is small compared to b in a certain sense, $\mathfrak{S} \sim \int |Q_b|^2 - \int |Q|^2 \sim b^2$ from (5.3). Can occur from (7.21), this forces b to decay.*

Proof: Multiply (7.18) by $\frac{\delta_1 b}{800}$ and sum with (7.20). We get

$$\begin{aligned} \left(\int \phi_A |\epsilon|^2 \right)_s + \frac{\delta_1 b}{800} (f_1)_s &\geq \frac{\delta_1^2 b}{800} \left(\int |\nabla \tilde{\epsilon}|^2 + \int |\tilde{\epsilon}|^2 e^{-|y|} \right) + \frac{b}{800} \int_A^{2A} |\epsilon|^2 \\ &\quad + \frac{c\delta_1 b}{1000} \Gamma_b - C \frac{\lambda^2}{b^2} E(u(t)) - \Gamma_b^{\frac{l}{2}} \int |\nabla \epsilon|^2, \end{aligned} \quad (7.24)$$

We first integrate the left-hand side of (7.24) by parts in time:

$$\begin{aligned} b(f_1)_s &= \left(b\tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) dv + b((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d)) \right)_s \\ &\quad - b_s((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d)), \end{aligned} \quad (7.25)$$

where \tilde{f}_1 given by (7.23). (7.25) now yields

$$\begin{aligned} &\left(\int \phi_A |\epsilon|^2 + \frac{\delta_1}{800} (b\tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) dv + b((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d)) \right)_s \\ &\geq \frac{\delta_1^2 b}{800} \left(\int |\nabla \tilde{\epsilon}|^2 + \int |\tilde{\epsilon}|^2 e^{-|y|} + \int_A^{2A} |\epsilon|^2 \right) + \frac{c\delta_1 b}{1000} \Gamma_b - C \frac{\lambda^2}{b^2} E(u(t)) \\ &\quad - \Gamma_b^{\frac{l}{2}} \int |\nabla \epsilon|^2 + \frac{\delta_1}{800} b_s((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d)). \end{aligned}$$

We now inject the expression of the L^2 -norm:

$$\int |\epsilon|^2 + \int |Q_b|^2 + 2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) = e^{-2at} \int |u_0|^2 dx,$$

but

$$\int \phi_A |\epsilon|^2 = \int |\epsilon|^2 - \int (1 - \phi_A) |\epsilon|^2,$$

we compute:

$$\begin{aligned} \left(\int \phi_A |\epsilon|^2 \right)_s &= - \left(\left(\int |Q_b|^2 - \int |Q|^2 \right) + 2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) + \int (1 - \phi_A) |\epsilon|^2 \right)_s \\ &\quad + \left(\int |u_0|^2 e^{-2at} \right)_s \\ &= - \left(\left(\int |Q_b|^2 - \int |Q|^2 \right) + 2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) + \int (1 - \phi_A) |\epsilon|^2 \right)_s \\ &\quad - 2a\lambda^2 e^{-2at} \|u_0\|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

Thus, we get

$$\begin{aligned} (-\mathfrak{S})_s &\geq \frac{\delta_1^2 b}{800} \left(\int |\nabla \tilde{\epsilon}|^2 + \int |\tilde{\epsilon}|^2 e^{-|y|} + \int_A^{2A} |\epsilon|^2 \right) + \frac{c\delta_1 b}{1000} \Gamma_b - C \frac{\lambda^2}{b^2} E(u(t)) \\ &\quad - \Gamma_b^{\frac{1}{2}} \int |\nabla \epsilon|^2 + \frac{\delta_1}{800} b_s ((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d)) - C_0 \lambda^2. \end{aligned} \quad (7.26)$$

We now have

$$\Gamma_b^{\frac{1}{2}} \int |\nabla \epsilon|^2 \leq \Gamma_b^{\frac{1}{2}} (\Gamma_b^{1-C\eta} + \int |\nabla \tilde{\epsilon}|^2),$$

from the assumption $l > C\eta$. Next, we estimate from (7.5):

$$\left| b_s ((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d)) \right| \leq \Gamma_b^{\frac{1}{2}-C\eta} \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} + C\lambda^2 E(u(t)) \right).$$

Injecting these estimates into (7.26) yields (7.21). This concludes the proof of the proposition. \square

Note that now from (7.21) we obtain:

$$(\mathfrak{S}(s))_s \leq -Cb\Gamma_b + C \frac{\lambda^2}{b^2} E(u(t)) \leq -\frac{1}{2} Cb\Gamma_b \leq 0.$$

7.3. Proof of the Bootstrap (Proposition 5.2). Let f_2 be defined by:

$$f_2 = \left(\int |Q_b|^2 - \int Q^2 \right) - \frac{\delta_1}{800} \left(b\tilde{f}_1(b) - \int_0^b \tilde{f}_1(v) dv \right)$$

it satisfies (using the smallness (7.16) of ζ in L^2)

$$\frac{d_0}{C} < \frac{df_2}{db^2} \Big|_{b^2=0} < Cd_0, \quad (7.27)$$

with d_0 defined by :

$$0 < \frac{d}{db^2} \left(\int |Q_b|^2 \right) \Big|_{b^2=0} = d_0 < +\infty. \quad (7.28)$$

Now as a consequence of the control (7.3) and of the coercivity of the linearized energy under the chosen set of orthogonality conditions, we have the bounds:

$$\left(\mathfrak{S}(s) - f_2(b(s)) \right) \begin{cases} \geq -\Gamma_b^{1-Cl} + \frac{1}{C} \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right) \\ \leq CA^2 \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right) + \Gamma_b^{1-Cl}. \end{cases} \quad (7.29)$$

Indeed,

$$\begin{aligned} \mathfrak{S}(s) - f_2(b(s)) &= 2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) + \int (1 - \phi_A) |\epsilon|^2 \\ &\quad - \frac{\delta_1 b}{800} \left((\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d) \right). \end{aligned} \quad (7.30)$$

From the estimates on $\tilde{\zeta}$ of Lemma 5.1, the choice of A , we have:

$$\begin{aligned} \left| (\epsilon_2, (\tilde{\zeta}_{re})_d) - (\epsilon_1, (\tilde{\zeta}_{im})_d) \right| &\leq \Gamma_b^{\frac{1}{2}-C\eta} \left(\int_0^A |\epsilon|^2 \right)^{\frac{1}{2}} \\ &\leq A^2 \Gamma_b^{1-C\eta} + C \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right) \\ &\leq \Gamma_b^{1-Cl} + C \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right). \end{aligned}$$

The other term in (7.30) is estimated from the expression of energy:

$$\begin{aligned} &2(\epsilon_1, \Sigma + b\Theta_d - Re(\Psi)) + 2(\epsilon_2, \Theta - b\Sigma_d - Im(\Psi)) \\ &= 2E(Q_b) - 2\lambda^2 E(u(t)) \\ &+ \int |\nabla \epsilon|^2 - \int \left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_1^2 \\ &- \int \left(\frac{4\Theta^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} \epsilon_2^2 \\ &- 8 \int \frac{\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \epsilon_1 \epsilon_2 - \frac{2}{2+\frac{4}{d}} \int F(\epsilon), \end{aligned}$$

which can be rewritten as :

$$\begin{aligned} &2(\epsilon_1, \Sigma) + 2(\epsilon_2, \Theta) + \int (1 - \phi_A) |\epsilon|^2 = (L_+ \epsilon_1, \epsilon_1) + (L_- \epsilon_2, \epsilon_2) - \int \phi_A |\epsilon|^2 \\ &+ 2(\epsilon_1, Re(\Psi)) + 2(\epsilon_2, Im(\Psi)) + 2E(Q_b) - 2\lambda^2 E(u(t)) \\ &- \int \left(\left(\frac{4\Sigma^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} - \left(\frac{4}{d} + 1 \right) Q^{\frac{4}{d}} \right) \epsilon_1^2 \\ &- \int \left(\left(\frac{4\Theta^2}{d|Q_b|^2} + 1 \right) |Q_b|^{\frac{4}{d}} - Q^{\frac{4}{d}} \right) \epsilon_2^2 \\ &- 8 \int \frac{\Sigma\Theta}{d|Q_b|^2} |Q_b|^{\frac{4}{d}} \epsilon_1 \epsilon_2 - \frac{2}{2+\frac{4}{d}} \int F(\epsilon). \end{aligned}$$

We first estimate:

$$\begin{aligned} &|(\epsilon_1, Re(\Psi))| + |(\epsilon_2, Im(\Psi))| + E(Q_b) + 2\lambda^2 |E(u(t))| \\ &\leq \Gamma_b^{1-Cl} + \Gamma_b^l \left(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \right). \end{aligned}$$

The cubic term $\int |F(\epsilon)|$ and the rest of the quadratic form are controlled by $\delta(\alpha_0)(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|}) + \Gamma_b^{1+l}$, we thus obtain:

$$\begin{aligned} & \left| \mathfrak{F}(s) - f_2(b(s)) - \left((L_+ \epsilon_1, \epsilon_1) + (L_- \epsilon_2, \epsilon_2) - \int (\phi_A) |\epsilon|^2 \right) \right| \\ & \leq \delta(\alpha_0)(\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|}) + \Gamma_b^{1-Cl}. \end{aligned}$$

The upper bound follows from:

$$\int (1 - \phi_A) |\epsilon|^2 \leq CA^2 \log A (\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|}). \quad (7.31)$$

For the lower bound, we use the elliptic estimate on $L = (L_+, L_-)$ (for the proof see Appendix D in [13]), this ends the proof of (7.29).

We are now in a position to prove the pointwise bound (5.19):

$$\int |\nabla \epsilon|^2 + \int |\epsilon|^2 e^{-|y|} \lesssim \Gamma_{b(s)}^{\frac{2}{3}}.$$

Let $s_2 \in [s_0, s^+]$, if $b_s(s_2) \leq 0$, then (5.19) follows directly from (7.14). If $b_s(s_2) > 0$, let $s_1 \in [s_0, s^+]$ be from s_2 such that $b_s(s_1) = 0$, then either s_1 is attained or $s_1 = s_0$. In both cases, we have using (3.4):

$$\int |\nabla \epsilon(s_1)|^2 + \int |\epsilon(s_1)|^2 e^{-|y|} \leq \Gamma_{b(s_1)}^{\frac{3}{4}}$$

and thus

$$\mathfrak{F}(s_1) - f_2(b(s_1)) \leq \Gamma_{b(s_1)}^{\frac{17}{24}}, \quad (7.32)$$

from (7.29) and for $l > 0$ is small enough. Moreover, $b_s \geq 0$ on $[s_1, s_2]$ and thus:

$$b(s_2) \geq b(s_1). \quad (7.33)$$

We now use the Lyapunov control (7.21) to derive:

$$\mathfrak{F}(s_2) \leq \mathfrak{F}(s_1),$$

we then inject (7.29), (7.33) and (7.32) to conclude:

$$\begin{aligned} f_2(b(s_2)) + \frac{1}{C} \left(\int |\nabla \epsilon(s_2)|^2 + \int |\epsilon(s_2)|^2 e^{-|y|} \right) & \leq \mathfrak{F}(s_2) + \Gamma_{b(s_2)}^{1-Cl} \\ & \leq f_2(b(s_1)) + \Gamma_{b(s_1)}^{1-Cl} + \Gamma_{b(s_2)}^{1-Cl} \leq f_2(b(s_1)) + 2\Gamma_{b(s_2)}^{\frac{2}{3}}. \end{aligned}$$

The monotonicity (7.27) of f_2 in b and (7.33) now imply:

$$\int |\nabla \epsilon(s_2)|^2 + \int |\epsilon(s_2)|^2 e^{-|y|} \lesssim \Gamma_{b(s_2)}^{\frac{2}{3}}$$

which implies that Equations (5.19) holds at s_2 . This concludes the proof of (5.19).

Now we are going to prove the upper bound on blowup rate:

From (7.14) we obtain:

$$b_s \geq -\Gamma_{b(s)}^{1-C\eta}. \quad (7.34)$$

In particular:

$$\left(e^{\frac{\pi}{2b(s)}}\right)_s \leq e^{\frac{\pi}{2b(s)}} \frac{\pi \Gamma^{1-C\eta}}{2b^2} \leq 1$$

as $\Gamma_b \sim e^{-\frac{\pi}{b}}$, and therefore

$$e^{\frac{\pi}{2b(s)}} \leq e^{\frac{\pi}{b(0)}} + s - s_0 \leq s,$$

thus from (7.34) and the value of s_0 (we take $s_0 = e^{\frac{5\pi}{9b(0)}}$).

Finally, $\forall s \in [s_0, s^+]$,

$$b(s) \geq \frac{\pi}{2 \log(s)}, \quad (7.35)$$

we now rewrite the estimate (7.5) using (5.19) as follows

$$\left| \frac{\lambda_s}{\lambda} + b \right| \leq \Gamma_b^{\frac{1}{4}}. \quad (7.36)$$

Thus:

$$\frac{b}{2} \leq -\frac{\lambda_s}{\lambda} \leq 2b.$$

We integrate this in time and get: $\forall s \in [s_0, s^+]$,

$$-\log(\lambda(s)) \geq -\log \lambda(s_0) + \frac{1}{2} \int_{s_0}^s b \geq -\log \lambda(s_0) + \frac{\pi}{4} \left(\frac{s}{\log s} - \frac{s_0}{\log s_0} \right).$$

Now from (3.2):

$$-\log(\lambda(0)) \geq e^{\frac{2\pi}{3b(s_0)}} = s_0^{\frac{4}{3}},$$

and thus

$$-\log(\lambda(s)) \geq -\frac{2}{3} \log(\lambda(0)) + \frac{\pi}{4} \frac{s}{\log(s)}, \quad \text{i.e.} \quad \lambda(s) \leq \lambda^{\frac{2}{3}}(0) e^{-\frac{\pi}{4} \frac{s}{\log s}}. \quad (7.37)$$

This also implies: $\forall s \in [s_0, s_2]$,

$$-\log(\lambda(s)) \geq \frac{\pi}{4} \frac{s}{\log(s)} \geq \sqrt{s} \quad (7.38)$$

and taking the log of this inequality yields

$$\log |\log(\lambda(s))| \geq \frac{1}{2} \log(s), \quad \text{i.e.} \quad b \geq \frac{\pi}{4 \log |\log \lambda(s)|} \quad (7.39)$$

using (7.35). Therefore (5.16) is proved.

Now we are going to prove the monotonicity of λ : We turn to the proof of (5.17) and (5.18). From (5.16), (7.5) and (5.19), there holds:

$$-\frac{\lambda_s}{\lambda} \geq \frac{b}{2}$$

and thus: $\forall s_1, s_2 \in [s_0, s^+]$,

$$-\log\left(\frac{\lambda(s_2)}{\lambda(s_1)}\right) \geq \frac{1}{2} \int_{s_1}^{s_2} b \geq \frac{1}{2} \int_{s_1}^{s_2} \frac{ds}{\log(s)}. \quad (7.40)$$

This prove (5.17). To prove (5.18), we let $[t_k, t_{k+1}]$ be a doubling time interval, then from (7.40):

$$\log(2) = -\log\left(\frac{\lambda(t_{k+1})}{\lambda(t_k)}\right) \geq \frac{1}{2} \int_{t_k}^{t_{k+1}} \frac{dt}{\lambda^2(t) \log(s(t))}$$

and thus

$$1 \geq \frac{C(t_{K+1} - t_k)}{\lambda^2(t_k) \log(s(t_{k+1}))} \geq \frac{C(t_{K+1} - t_k)}{\lambda^2(t_k) \log |\log(\lambda(t_{k+1}))|} \geq \frac{C(t_{K+1} - t_k)}{\lambda^2(t_k) \log k}$$

and (5.18) follows.

The upper bound on b is a direct consequence (5.3). The lower bound $b > 0$ follows from (7.39).

Now we prove the blowup in finite time. We observe from (7.38) that

$$T = \int_0^{+\infty} \lambda^2(s) ds \leq \lambda^{\frac{4}{3}}(0) (C + \int_2^{+\infty} e^{-\frac{\pi}{4} \frac{s}{\log s}} ds) < +\infty.$$

Moreover, from (5.19),

$$\|u(t)\|_{H^1(\mathbb{R}^d)} \sim \frac{1}{\lambda(t)},$$

and thus the local well-posedness theory in H^1 ensures $\lambda(t) \rightarrow 0$ as $t \rightarrow T$.

The convergence of the concentration point is a consequence of (7.6), (7.38) and (5.19) which imply:

$$\int_{s_0}^{+\infty} |x_s| ds \leq C \int_{s_0}^{+\infty} \lambda(s) ds < +\infty.$$

8. DETERMINATION OF THE BLOW-UP SPEED

In this part, we prove that the blow-up holds with the log-log speed.

Observe from (7.29), (5.19) and (5.16) that:

$$\frac{b^2(s)}{C} \leq \mathfrak{S}(s) \leq C b^2(s).$$

Together with (7.21), this implies:

$$(\mathfrak{S})_s \leq e^{-\frac{C}{\sqrt{\mathfrak{S}}}},$$

integrating this in time yields:

$$b(s) \leq C\sqrt{\mathfrak{S}} \leq \frac{C}{\log(s)}$$

for s large enough. Integrating now (7.36) in time, we conclude that

$$-\log(\lambda(s)) \leq C \int_{s_0}^s b + C \leq C \frac{s}{\log(s)}$$

for s large enough, and thus together with (7.39):

$$\frac{1}{C} \leq b \log |\log(\lambda)| \leq C. \quad (8.1)$$

Now

$$\begin{aligned} -(\lambda^2 \log |\log \lambda|)_t &= -\lambda \lambda_t \log |\log(\lambda)| \left(2 + \frac{1}{|\log \lambda| \log |\log \lambda|} \right) \\ &= -\left(\frac{\lambda_s}{\lambda} + b \right) \log |\log \lambda| \left(2 + \frac{1}{|\log \lambda| \log |\log \lambda|} \right) \\ &\quad + b \log |\log| \left(2 + \frac{1}{|\log \lambda| \log |\log \lambda|} \right). \end{aligned} \quad (8.2)$$

From (7.36)

$$\int_t^T \left| \left(\frac{\lambda_s}{\lambda} + b \right) \log |\log \lambda| \right| \lesssim \int_t^T (b^2 \log |\log \lambda|) dt. \quad (8.3)$$

Injecting this into (8.2) integrated from t to T and using (8.1), we conclude that

$$\frac{T-t}{C} \leq \lambda^2(t) \log |\log \lambda(t)| \leq C(T-t)$$

from which

$$\frac{1}{C} \left(\frac{T-t}{\log |\log(T-t)|} \right)^{\frac{1}{2}} \leq \lambda(t) \leq C \left(\frac{T-t}{\log |\log(T-t)|} \right)^{\frac{1}{2}} \quad (8.4)$$

for t close enough to T .

But

$$2b(T-t) = \lambda^2,$$

and thus:

$$\frac{1}{C} \frac{1}{\log |\log(T-t)|} \leq b(t) \leq C \frac{1}{\log |\log(T-t)|}.$$

From this we obtain:

$$\frac{1}{C} |\log(T-t)| \log |\log(T-t)| \leq s(t) \leq C |\log(T-t)| \log |\log(T-t)|.$$

This prove (1.4). \square

Remark 8.1. *Following the ideas in [12], we can also prove the existence of a L^2 -profile at blow-up point. More precisely there exists u^* in $L^2(\mathbb{R}^d)$ (L^2 -profile) such that:*

$$u(t, x) - \frac{1}{\lambda^{\frac{d}{2}}(t)} Q_{b(t)}(t, \frac{x - x(t)}{\lambda(t)}) e^{i\gamma(t)} \rightarrow u^* \text{ in } L^2(\mathbb{R}^d), \quad t \rightarrow T. \quad (8.5)$$

Proof of Corollary 3.1: Let $S(t)$ be the propagator for the linear equation:

$$i\partial_t u + \Delta u = 0, \quad (t, x) \in [0, \infty[\times \mathbb{R}^d.$$

The Cauchy problem for (1.1) with $u(0) = u_0 \in H^1(\mathbb{R}^d)$ is equivalent to the integral equation:

$$u(t) = S(t)u_0 + i \int_0^t S(t-s)(|u(s)|^{\frac{4}{d}}u(s) + iau)ds.$$

We know from Lemma 6.2: there exist $T(\|u_0\|_{H^1(\mathbb{R}^d)}) > 0$ such that:

$$\forall 0 \leq a \leq 1, \quad \|u\|_{L^\infty([0,T];H^1)} \leq 2 \|u_0\|_{H^1(\mathbb{R}^d)}.$$

Let u a solution for (1.1) and v solution for (1.2) we have:

$$\begin{aligned} u - v &= S(t)(u_0 - v_0) + i \int_0^t S(t-s)(|u(s)|^{\frac{4}{d}}u(s) - |v(s)|^{\frac{4}{d}}v(s))ds \\ &\quad + ia \int_0^t S(t-t')u(t')dt'. \end{aligned}$$

By Strichartz we obtain (see the proof of Lemma 6.2):

$$\begin{aligned} \|u - v\|_{L^\infty([0,T];H^1)} &\leq \|u_0 - v_0\|_{H^1(\mathbb{R}^d)} \\ &\quad + CT^\gamma (\|u\|_{L^\infty([0,T];H^1)}^p + \|v\|_{L^\infty([0,T];H^1)}^p) \|u - v\|_{L^\infty([0,T];H^1)} \\ &\quad + CaT \|u\|_{L^\infty([0,T];H^1)}. \end{aligned}$$

Thus for $T_1 = \text{Min}(T(\|u_0\|_{H^1(\mathbb{R}^d)}), T(\|v_0\|_{H^1(\mathbb{R}^d)}))$ we obtain $\forall 0 \leq t \leq T_1$:

$$\begin{aligned} \|u - v\|_{L^\infty([0,T];H^1)} &\leq \|u_0 - v_0\|_{H^1(\mathbb{R}^d)} \\ &\quad + CT^\gamma (\|u_0\|_{H^1}^p + \|v_0\|_{H^1}^p) \|u - v\|_{L^\infty([0,T];H^1)} \\ &\quad + CaT \|u\|_{L^\infty([0,T];H^1)}. \end{aligned}$$

Now for $T_2 = \frac{1}{2} \text{Min}(\text{Max}^{-\frac{1}{\gamma}}(\|u_0\|_{H^1}^p, \|v_0\|_{H^1}^p), T_1)$:

$$\begin{aligned} \|u - v\|_{L^\infty([0,t];H^1)} &\leq \|u_0 - v_0\|_{H^1(\mathbb{R}^d)} + \frac{1}{2} \|u - v\|_{L^\infty([0,t];H^1)} \\ &\quad + a, \end{aligned}$$

thus

$$\|u - v\|_{L^\infty([0,t];H^1)} \lesssim \|u_0 - v_0\|_{H^1(\mathbb{R}^d)} + a. \quad \forall 0 < t < T_2.$$

Thus the map $(a, \phi) \rightarrow u(\cdot, a, \phi)$ is continuous in $(0, u_0)$ from $\mathbb{R} \times$

$H^1(\mathbb{R}^d)$ to $C([0, T_2], H^1(\mathbb{R}^d))$. Since T_2 only depends on $\|u_0\|_{H^1(\mathbb{R}^d)}$, this continuity extends to any interval $[0, T]$ in the maximal interval of existence of u .

We know after a time t_0 closed to blow-up time of u with the initial data u_0 , that $u(t_0)$ verifies C.I, and by continuity $v(t_0)$ verifies also C.I (the conditions C.I are stable by a small perturbations in H^1), then we obtain from Theorem 3.1 the blow up of v with the initial data $v(t_0)$ for (1.1). Therefore the solution of (1.1) emanating from v_0 blows up in finite time in the log-log regime. \square

REFERENCES

- [1] H. Berestycki and P.-L. Lions. *Nonlinear scalar field equations. II. Existence of infinitely many solutions.* Arch. Rational Mech. Anal., 82(1983):347–375.
- [2] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003.
- [3] J. Colliander and P. Raphael. *Rough blowup solutions to the L^2 critical NLS.* Math. Ann., 345(2009):307–366.
- [4] G. Fibich and F. Merle. *Self-focusing on bounded domains.* Phys. D, 155(2001):132–158.
- [5] T. Hmidi and S. Keraani. *Blowup theory for the critical nonlinear Schrödinger equations revisited.* Int. Math. Res. Not., 46(2005):2815–2828.
- [6] T. Kato. *On nonlinear Schrödinger equations* Ann. Inst. H. Poincaré Phys. Théor., 46(1987):113–129.
- [7] M.K Kwong. *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in \mathbf{R}^n .* Arch. Rational Mech. Anal., 105(1989):243–266.
- [8] P.-L. Lions. *The concentration-compactness principle in the calculus of variations. The locally compact case. II.* Ann. Inst. H. Poincaré Anal. Non Linéaire, 1(1984):223–283.
- [9] F. Merle and P. Raphael. *Blow up dynamic and upper bound on the blow up rate for critical nonlinear Schrödinger equation.* In Journées “Équations aux Dérivées Partielles” (Forges-les-Eaux, 2002), pages Exp. No. XII, 5. Univ. Nantes, Nantes, 2002.
- [10] F. Merle and P. Raphael. *Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation.* Geom. Funct. Anal., 13(2003):591–642.
- [11] F. Merle and P. Raphael. *On universality of blow-up profile for L^2 critical nonlinear Schrödinger equation.* Invent. Math., 156(2004):565–672.
- [12] F. Merle and P. Raphael. *Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation.* Comm. Math. Phys., 253(2005):675–704.
- [13] F. Merle and P. Raphael. *On a sharp lower bound on the blow-up rate for the L^2 critical nonlinear Schrödinger equation.* J. Amer. Math. Soc., 19(2006):37–90 (electronic).
- [14] M. Ohta and G. Todorova. *Remarks on global existence and blowup for damped nonlinear Schrödinger equations.* Discrete Contin. Dyn. Syst., 23(2009):1313–1325.
- [15] F. Planchon and P. Raphaël. *Existence and stability of the log-log blow-up dynamics for the L^2 -critical nonlinear Schrödinger equation in a domain.* Ann. Henri Poincaré, 8(2007):1177–1219.

- [16] P. Raphael. *Stability of the log-log bound for blow up solutions to the critical non linear Schrödinger equation*. Math. Ann., 331(2005):577–609.
- [17] M. Tsutsumi. *Nonexistence of global solutions to the Cauchy problem for the damped nonlinear Schrödinger equations*. SIAM J. Math. Anal., 15(1984):357–366.
- [18] M.I. Weinstein. *Nonlinear Schrödinger equations and sharp interpolation estimates*. Comm. Math. Phys., 87(1982/83):567–576.

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